Oscillatory correlation of delayed random walks

Toru Ohira
Sony Computer Science Laboratory, 3-14-13 Higashi-gotanda, Shinagawa, Tokyo 141, Japan

(Received 8 May 1996; revised manuscript received 7 October 1996)

We investigate analytically and numerically the statistical properties of a random walk model with delayed transition probability dependence (delayed random walk). The characteristic feature of such a model is the oscillatory behavior of its correlation function. We investigate a model whose transient and stationary oscillatory behavior is analytically tractable. The correspondence of the model with a Langevin equation with delay is also considered. [S1063-651X(97)51402-4]

PACS number(s): 02.50.−r, 05.90.+m, 87.10.+e

Noise and correlative effects (memory) are two elements which are associated with many natural systems. In physics, two main approaches have been developed to study such systems with noise and memory. One approach is formulating the model in physical space with a differential equation of motion such as the ‘‘generalized Langevin equation’’ [1,2]. The other is to formulate a model in probability space as a non-Markovian problem as in the ‘‘generalized master equation’’ approach [3]. These two avenues have been developed and applied to various problems in physics. Examples include studies on the Alder-Wainwright effect [4], spin relaxation [5], and driven two-level atoms [6].

The delayed stochastic system we discuss here can be viewed as a special case, where only a single (memory) point at a fixed time interval in the past has influence on the current state of the system. Research of such systems, particularly those with no noise, has been carried out in fields of mathematics [7], biology [8], artificial neural networks [9], electrical circuits [10], as well as in physics [11]. Models with both noise and delay have also been considered numerically [12] and analytically as an extension of the Langevin equation [13]. These works represent approaches and formulations in physical space. For the probability space approach, ‘‘delayed random walk’’ is recently proposed [14] and has been applied to model human posture controls [15]. However, an analytical understanding of this random walk is yet far from being complete.

The main theme of this paper is to increase the analytical understanding of the behavior of a delayed random walk model. The oscillatory correlation function is found to be associated with delayed random walks [14,16]. We show here that such oscillatory behavior of the correlation function is analytically tractable. From the point of view of the study of random walks, this delayed random walk model provides an example whose correlation function behaves differently compared to commonly known random walks with memory, such as self-avoiding, or persistent walks [17]. In addition, we note that oscillatory or chaotic behavior associated with delays is generally difficult to analyze [12]. Hence, this model also serves as one of the rare analytically tractable examples among models with delay.

We consider a random walk which takes a unit step in a unit time. The delayed random walk we start with is an extension of a position dependent random walk whose step toward the origin is more likely when no delay exists. Formally, it has the following definition:

$$\begin{aligned} P(X_{t+1}=n;X_{t+1-\tau}=s) &= g(s-1)P(X_t=n-1;X_{t+1-\tau}=s;X_{t-\tau}=s-1) \\ &\quad + g(s+1)P(X_t=n-1;X_{t+1-\tau}=s;X_{t-\tau}=s+1) \\ &\quad + f(s-1)P(X_t=n+1;X_{t+1-\tau}=s;X_{t-\tau}=s-1) \\ &\quad + f(s+1)P(X_t=n+1;X_{t+1-\tau}=s;X_{t-\tau}=s+1), \end{aligned} \tag{1}$$

$$f(x)+g(x)=1, \tag{2}$$

where the position of the walker at time t is X_t, and $P(X_{t_1}=u_1;X_{t_2}=u_2)$ is the joint probability for the walker to be at u_1 and u_2 at time t_1 and t_2, respectively. $f(x)$ and $g(x)$ are transition probabilities to take a step to the negative and positive directions respectively at the position x. In this paper, we further place the conditions

$$f(x)>g(x) \quad (x>0), \quad f(-x)=g(x) \quad (\forall x). \tag{3}$$

These conditions make the delayed random walks symmetric with respect to the origin, which is attractive without delay $(\tau=0)$.

We now proceed to obtain a few properties from this general definition. By the symmetry with respect to the origin, the average position of the walker is 0. This symmetry is further used to inductively show [18] in the stationary state $(t\to\infty)$ that

$$P(X_{t+1}=n;X_t=n+1)=P(X_{t+1}=n+1;X_t=n). \tag{4}$$

We derived the stationary probability distribution for the previously discussed delayed random walk model using this property [14]. Also, the multiplication of Eq. (1) for the stationary state by $\cos(\alpha n)$ and summation over n and s yields for the generating function:

$$\langle\cos(\alpha X_t)\rangle=\cos(\alpha)\langle\cos(\alpha X_t)\rangle + \sin(\alpha)\langle\sin(\alpha X_t)\{f(X_{t-\tau})-g(X_{t-\tau})\}\rangle. \tag{5}$$

In particular, we have the following invariant relationship with respect to the delay:

$$\frac{1}{2}=\langle X_t\{f(X_{t-\tau})-g(X_{t-\tau})\}\rangle. \tag{6}$$

This invariant property is used below.

We will consider a specialized model for the rest of this paper [19]. We define $f(x)$ and $g(x)$ as

$$f(x)=\frac{1}{2}(1+2d)\quad(x>a),\quad \frac{1}{2}(1+\beta x)\quad(-a\leqslant x\leqslant a),\quad \frac{1}{2}(1-2d)\quad(x<-a),$$

$$g(x)=\frac{1}{2}(1-2d)\quad(x>a),\quad \frac{1}{2}(1-\beta x)\quad(-a\leqslant x\leqslant a),\quad \frac{1}{2}(1+2d)\quad(x<-a). \tag{7}$$

Physically, this model implies that when $\tau=0$ the transition probability for the walker to move toward the origin increases linearly at a rate of $\beta\equiv d/a$ as the distance increases from the origin up to the potition a after which the transition probability is held constant. We assume that with sufficiently large a, we can ignore the probability for the walker to be outside of the range $(-a,a)$.

Then, the previous invariant relation in Eq. (6) becomes the following with this model:

$$\langle X_tX_{t-\tau}\rangle=K(\tau)=\frac{1}{2\beta}. \tag{8}$$

This invariance with respect to τ of the correlation function with τ steps apart is a simple characteristic of this delayed random walk model. This property is a key to obtaining the analytical expression for the correlation function, to which we now turn our attention.

For the stationary state and $0\leqslant u\leqslant\tau$, the following is obtained from the definition (1).

$$\begin{aligned}P(X_t=n;X_{t-u}=l)&=\sum_s g(s)P(X_t=n-1;X_{t-(u-1)}=l;X_{t-\tau}=s)\\&\quad+\sum_s f(s)P(X_t=n+1;X_{t-(u-1)}=l;X_{t-\tau}=s).\end{aligned} \tag{9}$$

We can derive the following equation for the correlation function by multiplication of this equation by nl and summing over.

$$K(u)=K(u-1)-\beta K(\tau+1-u)\quad(0\leqslant u\leqslant\tau). \tag{10}$$

A similar argument gives for $\tau<u$,

$$K(u)=K(u-1)-\beta K(u-1-\tau)\quad(\tau<u). \tag{11}$$

Equations (10) and (11) can be solved explicitly using (8). In particular, for $0\leqslant u\leqslant\tau$ we obtain

$$K(u)=K(0)\frac{(m_+^u-m_+^{u-1})-(m_-^u-m_-^{u-1})}{m_+-m_-}-\frac{1}{2}\frac{(m_+^u-m_-^u)}{m_+-m_-},$$

$$K(0)=\frac{1}{2\beta}\frac{(m_+-m_-)+\beta(m_+^\tau-m_-^\tau)}{(m_+^\tau-m_+^{\tau-1})-(m_-^\tau-m_-^{\tau-1})}$$

$$m_\pm=\left(1-\frac{\beta^2}{2}\right)\pm\frac{\beta}{2}\sqrt{\beta^2-4}. \tag{12}$$

For $\tau<u$, it is possible to write $K(u)$ in a multiple summation form, though the expression becomes rather complex. For example, with $\tau<u\leqslant 2\tau$,

$$K(u)=\frac{1}{2\beta}-\beta\sum_{i=0}^{u-1-\tau}K(i) \tag{13}$$

where $K(i)$ summed is given by Eq. (12).

The behavior of the correlation function is shown in Fig. 1. As we increase τ, oscillatory behavior of the correlation function appears. The decay of the peak envelope is found numerically to be exponential. The decay rate of the envelope for small u is approximately $1/[2K(0)]$. Also we note the mean square postion $[K(0)]$ increases with increasing delay τ.

Analysis of the correlation function for the transient state can be done with a similar argument as in the stationary state. We can derive the set of coupled dynamical equations as follows:

$$K(0,t+1)=K(0,t)+1-2\beta K(\tau,t),$$

$$K(u,t+1)=K(u-1,t)-\beta K(\tau-(u-1),t+1-u)\quad(1\leqslant u\leqslant\tau)$$

$$K(u,t+1)=K(u-1,t)-\beta K((u-1)-\tau,t+1-u)\quad(u>\tau) \tag{14}$$

For the initial condition, we need to specify the correlation function for the interval of initial τ steps. Let us consider a random walk, which is held at the origin before it begins to take a step, thus performing a homogeneous random walk for the steps $(1,\tau)$. This translates to the initial condition for the correlation function as

$$K(u,t)=t-u\quad(0\leqslant u\leqslant\tau). \tag{15}$$

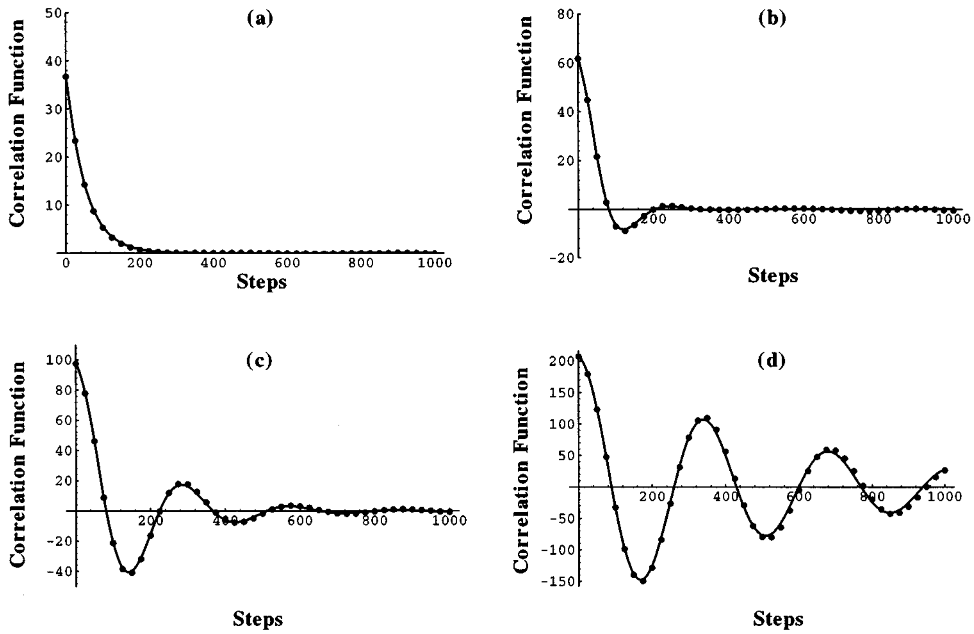

FIG. 1. Stationary correlation function $K(u)$ from simulations (dots) as a function of steps u with varying τ compared with the analytical solution obtained in the text (line). The parameters are set as $a=50$, $d=0.4$, and $\tau=$(a)10, (b)40, (c)60, (d)80. The simulation performed random walks of 6000 steps starting from the origin. The position data after 4500 steps are used to compute the correlation and averaged over 10 000 trials.

The solution can be iteratively generated for Eq. (14) given this initial condition. We have plotted some examples for the dynamics of the mean square displacement $K(0)$ in Fig. 2. Again, the oscillatory behavior arises with increasing τ. Hence, in the model discussed here the oscillatory behavior with increasing delay appears in both its stationary and transient states.

Let us now briefly discuss relationship of this model to the Langevin equation with delay:

$$\frac{d}{dt}X_t=-\beta X_{t-\tau}+\xi_t, \quad \langle \xi_{t_1}\xi_{t_2}\rangle=\delta(t_1-t_2). \tag{16}$$

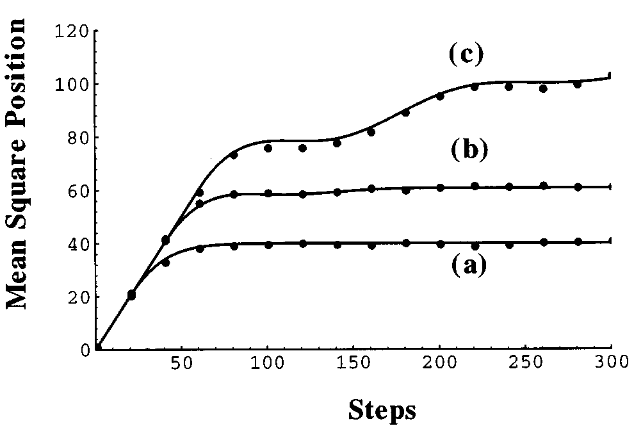

FIG. 2. Examples of dynamics of the mean square position $\langle X^2\rangle=K(0)$ with varying delay τ. The data are from simulations (dots) averaged over 10 000 trials, and from the analytical solutions (line). The parameters are set as $a=50$, $d=0.45$, and $\tau=$(a)20, (b)40, (c)60.

This Langevin equation is a special case of the equation considered in [13]. It should be noted that the equation is normalized with the "width" of the noise ξ_t. It has been shown that (1) the equation is stationary if and only if $\tau<\pi/2\beta$; and (2) the stationary correlation function $K(r)\equiv\langle X_t X_{t-r}\rangle$ has the following form when $r<\tau$:

$$K(r)=K(0)\cos(\beta r)-\frac{1}{2\beta}\sin(\beta r), \quad K(0)=\frac{1+\sin(\beta\tau)}{2\beta\cos(\beta\tau)} \tag{17}$$

When $\beta\ll 1$ (or $a\gg d$), the delayed random walk model approximately corresponds to this Langevin equation with delay. In particular, we can obtain Eq. (17) from the result (12) obtained for the delayed random walk, by expanding in small β.

Some points of discussion are now in order. The first point is how this model is placed in relation to other models with noise and delay (or memory, to be more general). In particular we note that the Langevin equation discussed here is not a special case of the generalized Langevin equation which is consistent with the fluctuation-dissipation theorem. As argued in [1] for the generalized Langevin equation, the noise term needs to be "colored" in Eq. (16) for consistency. Investigation of the colored noise case in its relation to delayed random walks as well as further studies of the correspondence of dynamical aspects of (1) and (16) are currently underway. Finally, we ask what the possible applications are of delayed random walks. As mentioned before, the model with a different transition property has been applied [14] to describe the qualitative statistical behavior of the center of gravity in a human posture control experiment [15]. Also, oscillatory correlation functions appear in such numerical studies of phase separation dynamics under stirring

[20] and of response dynamics of neural recepter cell syncytium [21]. Applications or relations to these and other systems are currently being sought and considered.

The model provided here is simple, yet has shown some characteristics of systems with noise and delay. It is hoped that further investigation of this and extended delayed random walks will provide us deeper understanding of delayed stochastic systems.

The author would like to thank Drs. Y. Okabe and K. Umeno for discussions, and J. Milton and M. Mackey for suggestions of references.

[1] R. Kubo, in *1965 Tokyo Summer Lectures in Theoretical Physics, Part I, Many-Body Theory* (Benjamin, New York, 1966), pp.1–16; R. Kubo, Rep. Prog. Phys. **29,** 255 (1966).
[2] H. Mori, Prog. Theor. Phys. **33,** 423 (1965).
[3] S. Nakajima, Prog. Theor. Phys. **20,** 948 (1958); R. Zwanzig, J. Chem. Phys. **33,** 1338 (1960).
[4] Y. Okabe, J. Stat. Phys. **45,** 953 (1986).
[5] M. A. Despósito and E. S. Hernández, J. Phys. A **28,** 775 (1995).
[6] J. R. Brinati, S. S. Mizrahi, and G. A. Prataviera, Phys. Rev. A **52,** 2804 (1995).
[7] K. L. Cooke and Z. Grossman, J. Math. Anal. Appl. **86,** 592 (1982).
[8] M. C. Mackey and L. Glass, Science **197,** 287 (1977); A. Longtin and J. Milton, Biol. Cybern. **61,** 51 (1989); J. Milton and A. Longtin, Vis. Res. **30,** 515 (1990).
[9] C. M. Marcus and R. M. Westervelt, Phys. Rev. A **39,** 347 (1989).
[10] J. Losson, M. C. Mackey, and A. Longtin, CHAOS **3,** 167 (1993).
[11] M. W. Derstine, H. M. Gibbs, F. A. Hopf, and D. L. Kaplan, Phys. Rev. A **26,** 3720 (1982); R. Vallée, C. Delisle, and J. Chrostowski, *ibid.* **30,** 336 (1984); K. Pyragas, Phys. Lett. A **170,** 421 (1992).
[12] J. Milton, A. Longtin, A. Beuter, M. C. Mackey, and L. Glass, J. Theor. Biol. **138,** 129 (1989); A. Longtin, J. G. Milton, J. Bos, and M. C. Mackey, Phys. Rev. A **41,** 6992 (1990); A. Longtin, *ibid.* **44,** 4801 (1991).
[13] U. Küchler and B. Mensch, Stoch. Stoch. Rep. **40,** 23 (1992).
[14] T. Ohira and J. G. Milton, Phys. Rev. E. **52,** 3277 (1995).
[15] J. J. Collins and C. J. DeLuca, Exp. Brain Res. **95,** 308 (1993); J. J. Collins and C. J. DeLuca, Phys. Rev. Lett. **73,** 764 (1994).
[16] T. Ohira, Sony Computer Science Laboratory Technical Report No. SCSL-TR-96-010, 1996 (unpublished).
[17] G. H. Weiss, *Aspects and Applications of the Random Walk* (North-Holland, Amsterdam, 1994).
[18] Details will be shown elsewhere; T. Ohira, Sony Computer Science Laboratory Technical Report No. SCSL-TR-96-015, 1996 (unpublished).
[19] This restriction for $f(x)$ and $g(x)$ to be linearly dependent on x is necessary for the following analytical treatment for the correlation function.
[20] This work shows an oscillation in the radial correlation function; A. M. Lacasta, J. M. Sancho, and F. Sagués, Phys. Rev. Lett. **75,** 1791 (1995).
[21] H. Funakubo, Y. Kashimori, and T. Kambara, in *Progress in Neural Information Processing: Proceedings of the International Conference on Neural Information Processings, Hong Kong* (Springer, Berlin, 1996), Vol. 1.